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О НУЛЯХ НЕКОТОРЫХ ФУНКЦИЙ, СВЯЗАННЫХ С ПЕРИОДИЧЕСКИМИ ДЗЕТА-ФУНКЦИЯМИ

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Аннотация

В статье полученно, что линейная комбинация периодической дзета-функции и периодической дзета-функции Гурвица и более общие комбинации этих функций имеют бесконечно много нулей, лежащих в правой стороне критической полосы.

Ключевые слова: нули аналитической функции, периодическая дзета-функция, периодическая дзета-функция Гурвица, универсальность.

ON THE ZEROS OF SOME FUNCTIONS RELATED TO PERIODIC ZETA-FUNCTIONS

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Abstract

In the paper, we obtain that a linear combination of the periodic and periodic Hurwitz zeta-functions, and more general combinations of these functions have infinitely many zeros lying in the right-hand side of the critical strip.

Keywords: periodic zeta-function, periodic Hurwitz zeta-function, universality, zeros of analytic function.

1. Introduction

Let $s = \sigma + it$ be a complex variable, and let $\zeta(s)$ and $\zeta(s, \alpha)$ with $0 < \alpha \leqslant 1$ denote the Riemann and Hurwitz zeta-functions, respectively. In this paper, we deal with generalizations of the functions $\zeta(s)$ and $\zeta(s, \alpha)$. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ and $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be two periodic sequences of complex numbers with minimal periods $k \in \mathbb{N}$ and $l \in \mathbb{N}$, respectively. The periodic zeta-function $\zeta(s; \mathfrak{a})$ and periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{b})$ are defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \qquad \text{and} \qquad \zeta(a, \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s},$$

and, in view of the equalities

$$\zeta(s; \mathfrak{a}) = \frac{1}{k^s} \sum_{m=1}^{k} a_m \zeta\left(s, \tfrac{m}{k}\right),$$

$$\zeta(s, \alpha; \mathfrak{b}) = \frac{1}{l^s} \sum_{m=0}^{l-1} b_m \zeta\left(s, \tfrac{m+\alpha}{l}\right),$$

which are valid for $\sigma > 1$, have analytic continuation to the whole complex plane, except for possible simple poles at the point $s = 1$. Clearly, $\zeta(s; \mathfrak{a}) = \zeta(s)$ for $a_m \equiv 1$, and $\zeta(s, \alpha; \mathfrak{b}) = \zeta(s, \alpha)$ for $b_m \equiv 1$.

The distribution of zeros of the function $\zeta(s; \mathfrak{a})$ was considered in [18], see also [20]. Define

$$c_{\mathfrak{a}} = \max(|a_m| :\ 1 \leqslant m \leqslant k), \qquad m_{\mathfrak{a}} = \min\{1 \leqslant m \leqslant k :\ a_m \neq 0\},$$

$$A(\mathfrak{a}) = \frac{m_{\mathfrak{a}} c_{\mathfrak{a}}}{|a_{m_{\mathfrak{a}}}|},$$

$$a_m^{\pm} = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} a_j \exp\left\{\pm 2\pi i j \tfrac{m}{k}\right\},$$

$$\mathfrak{a}^{\pm} = \left\{a_m^{\pm} :\ m \in \mathbb{N}\right\}$$

and

$$B(\mathfrak{a}) = \max\left\{A(\mathfrak{a}^{\pm})\right\}.$$

Then in [18], it was obtained that $\zeta(s; \mathfrak{a}) \neq 0$ for $\sigma > 1 + A(\mathfrak{a})$. Moreover, for $\sigma < -B(\mathfrak{a})$, the function $\zeta(s; \mathfrak{a})$ can only have zeros close to the negative real axis if $m_{\mathfrak{a}^+} = m_{\mathfrak{a}^-}$, and close to the straight line given by the equation

$$\sigma = 1 + \frac{\pi t}{\log \frac{m_{\mathfrak{a}^-}}{m_{\mathfrak{a}^+}}}$$

if $m_{\mathfrak{a}^+} \neq m_{\mathfrak{a}^-}$.

Denote by $\rho = \beta + i\gamma$ the zeros of the function $\zeta(s; \mathfrak{a})$. The zeros with $\beta < -B(\mathfrak{a})$ are called trivial. The number of trivial zeros ρ with $|\rho| \leqslant R$ is asymptotically equal to cR with some $c = c(\mathfrak{a}) > 0$. Other zeros of $\zeta(s; \mathfrak{a})$ are called non-trivial, and, by the above remarks, they lie in the strip $-B(\mathfrak{a}) \leqslant \sigma \leqslant 1 + A(\mathfrak{a})$.

Let $N(T; \mathfrak{a})$ be the number of non-trivial zeros ρ of $\zeta(s; \mathfrak{a})$ with $|\gamma| \leqslant T$. Then [18]

$$N(T; \mathfrak{a}) = \frac{T}{\pi} \log \frac{kT}{2\pi e m_{\mathfrak{a}} \sqrt{m_{\mathfrak{a}^-} m_{\mathfrak{a}^+}}} + O(\log T).$$

Moreover, the non-trivial zeros of $\zeta(s; \mathfrak{a})$ are clustered around the critical line $\sigma = \frac{1}{2}$.

In [15], it was obtained that the functions $F(\zeta(s; \mathfrak{a}))$ for some classes of operators F of the space of analytic functions have infinitely many zeros in the strip $\frac{1}{2} < \sigma < 1$.

The paper [2] is devoted to zeros of the function $\zeta(s, \alpha; \mathfrak{b})$. From properties of Dirichlet series, it follows that there exists $\sigma_1 > 0$ such that $\zeta(s, \alpha; \mathfrak{b}) \neq 0$ for $\sigma > \sigma_1$. For simplicity, suppose that $b_0 = 1$, and

$$q^{\pm}(m) = \sum_{k=0}^{l-1} b_k \exp\left\{\pm 2\pi i m \tfrac{\alpha+k}{l}\right\}.$$

Denote by $\rho(s, \hat{l})$ the distance of s from the line $\hat{l}$ on the complex plane, and let, for $\varepsilon > 0$,

$$L_\varepsilon(\hat{l}) = \left\{s \in \mathbb{C} : \rho(s, \hat{l}) < \varepsilon\right\}.$$

Then in [2], it is obtained that there exist constants $\sigma_0 < 0$ and $\varepsilon_0 > 0$ such that $\zeta(s, \alpha; \mathfrak{b}) \neq 0$ for $\sigma < \sigma_0$ and

$$s \notin L_{\varepsilon_0}\left((\sigma - 1)\log \tfrac{r_1}{r_2} - \pi t = \log\left|\tfrac{q^-(r_2)}{q^+(r_1)}\right|\right),$$

where $r_1 = \min\{m \in \mathbb{N} : q^+(m) \neq 0\}$ and $r_2 = \min\{m \in \mathbb{N} : q^- \neq 0\}$. Using the above result, non-trivial zeros of $\zeta(s, \alpha; \mathfrak{b})$ are defined. Namely, the zero $\rho = \beta + i\gamma$ of $\zeta(s, \alpha; \mathfrak{b})$ is called non-trivial if $\sigma_0 \leqslant \beta \leqslant \sigma_1$. The zero $\hat{\rho}$ is called trivial if

$$\hat{\rho} \in L_{\varepsilon_0}\left((\sigma - 1)\log \tfrac{r_1}{r_2} - \pi t = \log\left|\tfrac{q^-(r_2)}{q^+(r_1)}\right|\right),$$

It is known that the function $\zeta(s, \alpha; \mathfrak{b})$ has infinitely many trivial zeros.

Denote by $N(T, \alpha; \mathfrak{b})$ the number of non-trivial zeros ρ of the function $\zeta(s, \alpha; \mathfrak{b})$ with $|\gamma| \leqslant T$ according multiplicities. Then in [2], it was proved that

$$N(T, \alpha; \mathfrak{b}) = \frac{T}{\pi} \log \frac{Tk}{2\pi e \alpha} + O(\log T).$$

Moreover,

$$\sum_{|\gamma| < T} \left(\beta - \tfrac{1}{2}\right) = -\tfrac{T}{2\pi} \log \tfrac{k}{\alpha} + \tfrac{T}{2\pi}\left(\log\left|q^+(r_1)\right| + \log\left|q^-(r_2)\right|\right) + O(\log T).$$

The latter formula shows that the non-trivial zeros of the function $\zeta(s,\alpha;\mathfrak{b})$ are clustered around the line $\sigma=\frac{1}{2}$.

The aim of this paper is to show that the function $\zeta(s,\alpha;\mathfrak{b})$ with some, for example,transcendental parameter α, and some combinations of the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ have infinitely many zeros in the strip $D=\left\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\right\}$. Denote by $A_T(\sigma_1,\sigma_2,c)$ the assertion that, for any σ_1,σ_2, $\frac{1}{2}<\sigma_1<\sigma_2<1$, there exists a constant $c=c(\sigma_1,\sigma_2,f)>0$ such that, for sufficiently large T, the function $f(s)$ has more than cT zeros in the rectangle

$$\sigma_1<\sigma<\sigma_2, \quad 0<t<T.$$

Let

$$L(\alpha)=\{\log(m+\alpha): m\in\mathbb{N}_0\}.$$

THEOREM 1. *Suppose that the set $L(\alpha)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Then, for the function $\zeta(s,\alpha;\mathfrak{b})$, the assertion $A_T(\sigma_1,\sigma_2,c)$ is true.*

Define the function

$$\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})=c_1\zeta(s;\mathfrak{a})+c_2\zeta(s,\alpha;\mathfrak{b}), \quad c_1,c_2\in\mathbb{C}\setminus\{0\}.$$

THEOREM 2. *Suppose that the number α is transcendental, the sequence $\mathfrak{a}$ is multiplicative, and, for each prime p, the inequality*

$$\sum_{m=1}^{\infty}\frac{|a_{p^m}|}{p^{\frac{\sigma}{2}}}\leqslant c<1 \tag{1}$$

is satisfied. Then, for the function $\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})$, the assertion $A_T(\sigma_1,\sigma_2,c)$ is true.

The next theorem is devoted to zeros of more general composite functions of $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$. We recall that $D=\left\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\right\}$. Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta, and $H^2(D)=H(D)\times H(D)$. Let $\beta_1>0$ and $\beta_2>0$. We say that the operator $F: H^2(D)\to H(D)$ belongs to the class $Lip(\beta_1,\beta_2)$ if it satisfies the following hypotheses:

1° For each polynomial $p=p(s)$, and any compact subset $K\subset D$ with connected complement, there exists an element $(g_1,g_2)\in F^{-1}\{p\}\subset H^2(D)$ such that $g_1(s)\neq 0$ on K;

2° For any compact subset $K\subset D$ with connected complement, there exist a positive constant c, and compact subsets K_1,K_2 of D with connected complements such that

$$\sup_{s\in K}|F(g_{11}(s),g_{12}(s))-F(g_{21}(s),g_{22}(s))|\leqslant c\sup_{1\leqslant j\leqslant 2}\sup_{s\in K_j}|g_{1j}(s)-g_{2j}(s)|^{\beta_j}$$

for all $(g_{r1},g_{r2})\in H^2(D)$, $r=1,2$.

THEOREM 3. *Suppose that the number α is transcendental, the sequence $\mathfrak{a}$ is multiplicative, inequality* (1) *is satisfied and $F \in Lip(\beta_1, \beta_2)$. Then, for the function $F(\zeta(s; \mathfrak{a}), \zeta(s, \alpha; \mathfrak{b}))$, the assertion $A_T(\sigma_1, \sigma_2, c)$ is true.*

We note that the class $Lip(\beta_1, \beta_2)$ is not empty. For example, in [6] it is proved that the operator $F : H^2(D) \to H(D)$,

$$F(g_1, g_2) = c_1 g_1^{(k_1)} + c_2 g_2^{(k_2)},$$

where $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $k_1, k_2 \in \mathbb{N}$ and $g^{(k)}$ denotes the kth derivative of g, belongs to the class $Lip(1, 1)$. To prove this, it suffices to apply the integral Cauchy formula.

2. Lemmas

Proof of Theorems 1 - 3 are based on universality theorems for the corresponding functions, and the classical Rouché theorem. We remind that the universality of zeta-functions was discovered by S. M. Voronin who proved [21] an universality theorem for the Riemann zeta-function. For brevity, we denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, by $H_0(K)$, $K \in \mathcal{K}$, the class of non-vanishing continuous functions on K which are analytic in the interior of K, and by $H(K)$, $K \in \mathcal{K}$, the class of continuous functions on K which are analytic in the interior of K. Let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the latest version of the Voronin theorem is the following assertion, see, for example, [8].

LEMMA 1. *Suppose that $K \in \mathcal{K}$, and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\varliminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

The majority of other zeta and L-functions, among them the periodic zeta-function, [14], [5], the Hurwitz zeta-function with transcendental [10] or rational parameter [3], [1], the periodic Hurwitz zeta-function with transcendental parameter [4], zeta-functions of cusp forms [12], [13], L-functions from the Selberg class [19], [16], and others are universal in the Voronin sense. We state universality theorems for periodic and periodic Hurwitz zeta-functions.

LEMMA 2. *Suppose that the sequence $\mathfrak{a}$ is multiplicative and inequality* (1) *is satisfied. Let $K \in \mathcal{K}$, and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\varliminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau; \mathfrak{a}) - f(s)| < \varepsilon \right\} > 0.$$

Proof of the lemma is given in [14].

LEMMA 3. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$, and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau, \alpha; \mathfrak{b}) - f(s)| < \varepsilon\right\} > 0.$$

The lemma with transcendental parameter α has been obtained in [4], and, under hypotheses of the lemma, has been proved in [11].

In universality theory of zeta-functions, an important role is played by joint universality theorems when a collection of given analytic functions is approximated simultaneously by shifts of a collection of zeta-functions. The first joint universality result also was obtained by S. M. Voronin. In [22], investigating the functional independence of Dirichlet L-functions, he first of all infact obtained their joint universality. We remind a modern version of the Voronin theorem, see, for example, [9].

LEMMA 4. *Suppose that $\chi_1, \dots, \chi_r$ be pairwise non-equivalent Dirichlet characters, and $L(s,\chi_1), \dots, L(s,\chi_r)$ be the corresponding Dirichlet L-functions. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$, and $f_j(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(s+i\tau, \chi_j) - f_j(s)| < \varepsilon\right\} > 0.$$

The joint universality of the periodic zeta-function and the periodic Hurwitz zeta-function has been considered in [6], and the following assertion has been proved.

LEMMA 5. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, inequality (1) is satisfied, and the number α is transcendental. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$ and $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau; \mathfrak{a}) - f_1(s)| < \varepsilon\,,\right.$$

$$\left.\sup_{s\in K_2} |\zeta(s+i\tau, \alpha; \mathfrak{b}) - f_2(s)| < \varepsilon\right\} > 0.$$

Now we state a generalization of Lemma 5 from the paper [7].

LEMMA 6. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, inequality (1) is satisfied, the number α is transcendental, and that $F \in Lip(\beta_1, \beta_2)$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |F\left(\zeta(s+i\tau; \mathfrak{a}), \zeta(s+i\tau, \alpha; \mathfrak{b})\right) - f(s)| < \varepsilon\right\} > 0.$$

For the proof of theorems on the number of zeros of zeta-functions and their certain combinations, the classical Rouché theorem is useful. For convenience, we state this theorem as a separate lemma.

LEMMA 7. *Let the functions $g_1(s)$ and $g_2(s)$ are analytic in the interior of a closed contour L and on L, and let on L the inequalities $g_1(s) \neq 0$ and $|g_2(s)| < |g_1(s)|$ be satisfied. Then the functions $g_1(s)$ and $g_1(s)+g_2(s)$ have the same number of zeros in the interior of L.*

Proof of the lemma can be found, for example, in [17].

3. Proofs of theorems

Proof of Theorem 1. Let

$$\sigma_0 = \frac{\sigma_1+\sigma_2}{2}, \qquad r = \frac{\sigma_2-\sigma_1}{2},$$

and let the number $\varepsilon > 0$ satisfy the inequality

$$\varepsilon < \frac{1}{10} \min_{|s-\sigma_0|=r} |s-\sigma_0| = \frac{r}{10}. \tag{2}$$

Suppose that $\tau \in \mathbb{R}$ satisfies the inequality

$$\sup_{|s-\sigma_0|\leqslant r} |\zeta(s+i\tau, \alpha; \mathfrak{b}) - (s-\sigma_0)| < \varepsilon. \tag{3}$$

Then, in view of (2), we have that the functions $\zeta(s+i\tau,\alpha;\mathfrak{b}) - (s-\sigma_0)$ and $s-\sigma_0$ in the disc $|s-\sigma_0| \leqslant r$ satisfy the hypotheses of Lemma 7. Hence, the function $\zeta(s,\alpha;\mathfrak{b})$ has a zero in the disc $|s-\sigma_0| \leqslant r$. Since, by Lemma 3, the set of τ satisfying inequality (3) has a positive lower density, we obtain that there exists a constant $c = c(\sigma_1, \sigma_2, \alpha, \mathfrak{b}) > 0$ such that for the function $\zeta(s,\alpha;\mathfrak{b})$ the assertion $A_T(\sigma_1,\sigma_2,c)$ is true. □

Proof of Theorem 2. We preserve the notation for σ_0 and r, and take in Lemma 5

$$f_1(s) = \varepsilon, \qquad f_2(s) = \frac{1}{c_2}(s-\sigma_0),$$

where the positive number ε satisfies the inequality

$$(|c_1|+|c_2|)\varepsilon < \frac{1}{10} \min_{|s-\sigma_0|=r} |s-\sigma_0| = \frac{r}{10}. \tag{4}$$

Suppose that $\tau \in \mathbb{R}$ satisfies the inequalities

$$\sup_{|s-\sigma_0|\leqslant r} |\zeta(s+i\tau;\mathfrak{a}) - f_1(s)| < \varepsilon \tag{5}$$

and

$$\sup_{|s-\sigma_0|\leqslant r} |\zeta(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon. \tag{6}$$

Then, for these τ, we have that

$$\sup_{|s-\sigma_0|\leqslant r} |(c_1\zeta(s+i\tau;\mathfrak{a}) + c_2\zeta(s+i\tau,\alpha;\mathfrak{b})) - (c_1 f_1(s) + c_2 f_2(s))| \\ < 2(|c_1| + |c_2|)\varepsilon.$$

Moreover, by the definition of $f_1(s)$ and $f_2(s)$,

$$\sup_{|s-\sigma_0|\leqslant r} |c_1 f_1(s) + c_2 f_2(s) - (s-\sigma_0)| = |c_1|\varepsilon.$$

Therefore,

$$\sup_{|s-\sigma_0|=\rho} |(c_1\zeta(s+i\tau;\mathfrak{a}) + c_2\zeta(s+i\tau,\alpha;\mathfrak{b})) - (s-\sigma_0)| < 3(|c_1| + |c_2|)\varepsilon.$$

This and (4) show that the functions

$$c_1\zeta(s+i\tau;\mathfrak{a}) + c_2\zeta(s+i\tau,\alpha;\mathfrak{b})) - (s-\sigma_0)$$

and $s-\sigma_0$ on the disc $|s-\sigma_0| \leqslant r$ satisfy the hypotheses of Lemma 7. Therefore, the function $c_1\zeta(s+i\tau;\mathfrak{a}) + c_2\zeta(s+i\tau,\alpha;\mathfrak{b}))$ has a zero in the disc $|s-\sigma_0| \leqslant r$. However, by Lemma 5, the set of τ satisfying inequalities (5) and (6) has a positive lower density. Hence, there exists a constant $c = c(\sigma_1, \sigma_2, \alpha, \mathfrak{a}, \mathfrak{b}) > 0$ such that, for the function $c_1\zeta(s+i\tau;\mathfrak{a}) + c_2\zeta(s+i\tau,\alpha;\mathfrak{b}))$, the assertion $A_T(\sigma_1, \sigma_2, c)$ is valid. □

Proof of Theorem 3. We argue similarly as above. Suppose that $\tau \in \mathbb{R}$ satisfies the inequality

$$\sup_{|s-\sigma_0|\leqslant r} |F(\zeta(s+i\tau;\mathfrak{a}), \zeta(s+i\tau,\alpha;\mathfrak{b})) - (s-\sigma_0)| < \varepsilon. \tag{7}$$

and ε satisfies (2). Then the functions

$$F(\zeta(s+i\tau;\mathfrak{a}), \zeta(s+i\tau,\alpha;\mathfrak{b})) - (s-\sigma_0)$$

and $s-\sigma_0$ in the disc $|s-\sigma_0| \leqslant r$ satisfy the hypotheses of Lemma 7. Therefore, the function $F(\zeta(s+i\tau;\mathfrak{a}), \zeta(s+i\tau,\alpha;\mathfrak{b}))$ has a zero in the disc $|s-\sigma_0| \leqslant r$. However, in view of Lemma 6, the set of τ satisfying inequality (7) has a positive lower density. Thus, there exists a constant $c = c(\sigma_1, \sigma_2, \alpha, \mathfrak{a}, \mathfrak{b}, F) > 0$ such that, for the function $F(\zeta(s+i\tau;\mathfrak{a}), \zeta(s+i\tau,\alpha;\mathfrak{b}))$, the assertion $A_T(\sigma_1, \sigma_2, c)$ is valid. □

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