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DIOPHANTINE APPROXIMATION ON THE CURVES WITH NON–MONOTONIC ERROR FUNCTION IN THE p-ADIC CASE

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Аннотация

It is shown that a normal (according to Mahler) curve in $\mathbb{Z}_p^n$ satisfies a convergent Khintchine Theorem with a non-monotonic error function.

Let $\mathcal{F}_n$ be the set of the normal functions

$$a_n f_n(x) + \ldots + a_2 f_2(x) + a_1 f_1(x) + a_0,$$

with $n \geqslant 2$, $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$, and $f_i : \mathbb{Z}_p \to \mathbb{Z}_p$, $i = 1, \ldots, n$, be the normal functions. For $F \in \mathcal{F}_n$ define the height of F as $H = H(F) = \max_{0 \leqslant j \leqslant n} |a_j|$. Without loss of generality we will assume that $f_1(x) = x$ and $a_n = H$.

Let p be a prime number and $\mathbb{Q}_p$ is the complete field of the p-adic numbers. Let $\mu_\infty(U_\infty)$ be the Lebesgue measure of a measurable set $U_\infty \subset \mathbb{R}$. Denote by μ_p the normalized Haar measure on $\mathbb{Q}_p$ such that $\mu_p(\mathbb{Z}_p) = 1$. According to Mahler [1], the function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is called a normal if f has the form

$$f(x) = \sum_{n=0}^{\infty} \alpha_n (x - \alpha)^n,$$

where $|\alpha|_p \leqslant 1$, $|\alpha_n|_p \leqslant 1$ for all n and $\lim_{n\to\infty} |\alpha_n|_p = 0$. Moreover, the normal function over $\mathbb{Z}_p$ can be decomposed into the Taylor series [2].

Let P_n be a set of polynomials $P \in \mathbb{Z}[x]$ of degree $\leqslant n$ and $\Psi : \mathbb{N} \to \mathbb{R}^+$ be a function. Given a polynomial P, $H(P)$ will denote the height of P, i.e. the maximum of the absolute values of its coefficients. Let $W_n(\Psi)$ be a set of $x \in \mathbb{R}$ such that there are infinitely many $P \in P_n$ satisfying

$$|P(x)| < \Psi(H(P)).$$

In 1924 Khintchine [3] proved that if the sum $\sum_{H=1}^{\infty} \Psi(H)$ converges, then $\mu(W_1(\Psi))$ is null, while if the sum diverges and Ψ is decreasing, $\mu(\mathbb{R} \setminus W_1(\Psi))$ is null. In 1969

Sprindzuk proved that $\mu_\infty(W_n(\Psi)) = 0$ if $\Psi(H) = H^{-w}$ and $w > n$ (see [4]). Baker [5] has improved Sprindzuk's result and shown that $\mu_\infty(W_n(\Psi)) = 0$ if Ψ is monotonic function and

$$\sum_{H=1}^{\infty} \Psi(H)^{\frac{1}{n}} < \infty. \tag{1}$$

In the same paper Baker conjectured that condition (1) can be replaced by the convergence of $\sum_{H=1}^{\infty} H^{n-1}\Psi(H)$. This conjecture was proved by Bernik [6] in 1989. The divergence case was proved by Beresnevich [8] in 1999 who showed that $\mu_\infty(\mathbb{R} \setminus W_n(\Psi)) = 0$ if $\sum_{H=1}^{\infty} H^{n-1}\Psi(H) = \infty$.

In all of the aforementioned results it was assumed that Ψ was monotonic. In 2005 Beresnevich [7] showed that the monotonicity restriction can be avoid in the convergence case of Baker's conjecture. The result of Beresnevich was generalized to non-degenerate curves in $\mathbb{R}^n$ [9].

Our main result below is a convergent Khintchine Theorem without monotonicity condition for the normal curve in $\mathbb{Z}_p^n$.

Теорема 1. *Let $\Psi : \mathbb{N} \to \mathbb{R}^+$ be an arbitrary function (not necessarily monotonic) such that the sum $\sum_{h=1}^{\infty} h^n\Psi(h)$ converges. Let $L_n(\Psi)$ be the set of $x \in \mathbb{Z}_p$ such that there are infinitely many $F \in \mathcal{F}_n$ satisfying*

$$|F(x)|_p < \Psi(H(F)). \tag{2}$$

Then $\mu_p(L_n(\Psi)) = 0$.

Since the sum $\sum_{h=1}^{\infty} h^n\Psi(h)$ converges then $H^n\Psi(H)$ tends to 0 as $H \to \infty$. Hence $H^n\Psi(H) = o(1)$ and $\Psi(H) = o(H^{-n})$.

The set $L_n(\Psi)$ can be considered as the union of finite or countable number of the discs K_s. Further we fix one of them K_0. Without loss of generality we will assume that

$$radius(K_0) < p^{-n}. \tag{3}$$

To prove the theorem two different cases concerning the size of $|F'(x)|_p$ are considered. If $x \in L_n(\Psi)$ then x must satisfy at least one of these cases infinitely often. To prove that we will show that each set of x satisfying one of the conditions infinitely often has measure zero.

Case I. First the case of very small derivative is deal with.

Lemma 1. *The set of points $x \in K_0$ which satisfy*

$$|F(x)|_p < \Psi(H), \ \ |F'(x)|_p < H^{-1-v}, \ v > 0,$$

for infinitely many $F \in \mathcal{F}_n$ has measure zero.

This is proved using Theorem 1.5 from [10]. Using the notation in that theorem choose $T_0 = \ldots = T_n = H$, $K_p = H^{-1-v}$ and $\delta = H^{-n}$.

Proposition 1. *Let* $\rho = \min\{n, v/(n+1)\}$. *Let* $K_0 \subset \mathbb{Q}_p$ *and* $\mathbf{f} = (f_1, \dots, f_n) : K_0 \to \mathbb{Q}_p^n$ *be analytic non-degenerate map. For any* $x \in K_0$, *one can find a neighborhood* $\mathbb{V} \subseteq K_0$ *of* x *and* $\lambda > 0$ *with the following property: if* $B_p \subseteq \mathbb{V}$ *any ball then there exists* $E > 0$ *such that the set*

$$\cup_{F \in \mathcal{F}_n,\, 0 < H(F) \leqslant H} \left\{ x \in B_p : \, |F(x)|_p < H^{-n}, \; |F'(x)|_p < H^{-1-v} \right\}$$

has measure at most $EH^{-\rho\lambda}\mu_p(B_p)$.

For a non–negative integer t and for any $v > 0$, we denote by $\mathcal{A}(t)$ the set of $x \in B_p$ such that the system of inequalities

$$|F(x)|_p \ll H^{-n}, \;\; |F'(x)|_p < H^{-1-v} \tag{4}$$

holds for some $F \in \mathcal{F}_n$ with $2^{t-1} \leqslant H(F) < 2^t$. According to Proposition 1, $\mu_p(\mathcal{A}(t)) \ll 2^{\frac{-v\lambda t}{n+1}}$ with $v, \lambda > 0$. The set of $x \in K_0$ for which there are infinitely many $F \in \mathcal{F}_n$ satisfying (4) consists of points $x \in B_p$ which belong to infinitely many sets $\mathcal{A}(t)$. The sum $\sum_{t=1}^{\infty} \mu_p(\mathcal{A}(t))$ converges for $v > 0$ and the Borel–Cantelli Lemma can be used to complete the proof of the lemma.

Case II. Now we consider the set $L_n'(\Psi)$ of $x \in K_0$ such that there are infinitely many F satisfying $|F(x)|_p < \Psi(H)$ and $|F'(x)|_p > H^{-1-v}$.

Since the ring $\mathbb{Z}_p$ of p-adic integers is compact then for $F \in \mathcal{F}_n$ we may define a point $\alpha_F \in L_n'(\Psi)$ such that

$$|F'(\alpha_F)|_p = \min_{x \in L_n'(\Psi)} |F'(x)|_p.$$

Develop every function $F \in \mathcal{F}_n$ as a Taylor series so that

$$F(x) - F(\alpha_F) = (x - \alpha_F)\left(F'(\alpha_F) + \sum_{m=2}^{\infty} (m!)^{-1} F^{(m)}(\alpha_F)(x-\alpha_F)^{m-1} \right).$$

Using (3) and the fact that $|F'(x)|_p > H^{-1-v}$, we obtain

$$|F(x) - F(\alpha_F)|_p = |F'(\alpha_F)|_p |x - \alpha_F|_p,$$

and from (2)

$$|x - \alpha_F|_p \leqslant \Psi(H)|F'(\alpha_F)|_p^{-1}. \tag{5}$$

Further the proof of theorem depends on the range for $|F'(\alpha_F)|_p$.

Type 1. For $F \in \mathcal{F}_n$, let $\sigma(F)$ be the set of $x \in L_n'(\Psi)$ which satisfy

$$|F(x)|_p < \Psi(H), |F'(\alpha_F)|_p \geqslant H^{-1/2}. \tag{6}$$

Given a $F \in \mathcal{F}_n$ define the disc

$$\sigma_1(F) = \{x \in \mathbb{Z}_p : \, |x - \alpha_F|_p \leqslant (2p^{\frac{1}{p-1}} H |F'(\alpha_F)|_p)^{-1}\}. \tag{7}$$

If H is sufficiently large then by (5)

$$\mu_p(\sigma(F)) \ll H\Psi(H)\mu_p(\sigma_1(F)). \tag{8}$$

Fix any function F for which $\sigma(F) \neq 0$. Then we estimate the value of p-adic norm of function F in any point $x \in \sigma_1(F)$ by developing F as a Taylor series:

$$F(x) = F(\alpha_F) + F'(\alpha_F)(x-\alpha_F) + \sum_{m=2}^{\infty} \frac{1}{m!} F^{(m)}(\alpha_F)(x-\alpha_F)^m. \tag{9}$$

By (6), for $n \geqslant 2$ we have

$$|F(\alpha_F)|_p < \Psi(H) \ll H^{-2} \leqslant (2H)^{-1} \tag{10}$$

for sufficiently large H. Then by (7), we have

$$|F'(\alpha_F)(x-\alpha_F)|_p \leqslant |F'(\alpha_F)|_p (2p^{\frac{1}{p-1}} H |F'(\alpha_F)|_p)^{-1} < (2H)^{-1}. \tag{11}$$

For $m \geqslant 2$ using the inequalities $|m!|_p \leqslant p^{m/(p-1)}$ (see [11]), $|F^{(m)}(\alpha_F)|_p \leqslant 1$, (6) and (7) we obtain

$$\left| \frac{1}{m!} F^{(m)}(\alpha_F)(x-\alpha_F)^m \right|_p \leqslant p^{m/(p-1)} (2p^{\frac{1}{p-1}} H |F'(\alpha_F)|_p)^{-m} < (2H^{-1}H^{1/2})^m < (2H)^{-1}.$$

Using (9) – (11) and the previous inequality, we obtain

$$|F(x)|_p < (2H)^{-1} \tag{12}$$

for any $x \in \sigma_1(F)$.

Fix the vector $\mathbf{b} = (H, a_{n-1} \dots, a_2, a_1)$ and let the subclass of $\mathcal{F}_n$ of functions with the same vector $\mathbf{b}$ be denoted by $\mathcal{F}_n(\mathbf{b})$. The number of different $\mathcal{F}_n(\mathbf{b})$ is $\ll H^{n-1}$. Let $F_1, F_2 \in \mathcal{F}_n(\mathbf{b})$, and assume that they have different coefficients a_0. The number of different $\mathcal{F}_n(\mathbf{b})$ is $\ll H^{n-1}$. Let $F_1, F_2 \in \mathcal{F}_n(\mathbf{b})$, and assume that they have different coefficients a_0. Assume that there is an $x \in \sigma_1(F_1) \cap \sigma_1(F_2)$. Then by (12), $|F_1(x) - F_2(x)|_p < (2H)^{-1}$. On the other $F_1(x) - F_2(x)$ is an integer not greater than $2H$ in absolute value. Therefore, $|F_1(x) - F_2(x)|_p \geqslant 1/|F_1(x) - F_2(x)|_p > (2H)^{-1}$ that leads to a contradiction. Hence there is no such an x and $\sigma_1(F_1) \cap \sigma_1(F_2) = 0$. Hence,

$$\sum_{F \in \mathcal{F}_n(\mathbf{b})} \sigma_1(F) \leqslant \mu_p(K_0) \tag{13}$$

Together with (8) this gives

$$\sum_{F \in \mathcal{F}_n(\mathbf{b})} \mu_p(\sigma(F)) \ll H\Psi(H)\mu_p(K_0).$$

Summing this over all vectors $\mathbf{b}$ gives

$$\sum_{H=1}^{\infty}\sum_{\mathbf{b}}\sum_{F\in\mathcal{F}_n(\mathbf{b})}\mu_p(\sigma(F))\ll\sum_{H=1}^{\infty}H^n\Psi(H)\mu_p(K_0)<\infty.$$

The Borel–Cantelli lemma can now be used to complete the proof.

Type 2. For $F\in\mathcal{F}_n$, let $\sigma(F)$ be the set of $x\in L_n'(\Psi)$ which satisfy

$$|F(x)|_p<\Psi(H), H^{-1-v}<|F'(\alpha_F)|_p<H^{-1/2},\quad 0<v<1/4. \tag{14}$$

Define the set $\sigma_2(F)\supset\sigma(F)$ as the set of points x which satisfy the inequality

$$|x-\alpha_F|_p\leqslant p^{\frac{1}{1-p}}H^{-2}|F'(\alpha_F)|_p^{-1}$$

for $\alpha_F\in\sigma(F)$. It is clear that

$$\mu_p(\sigma_2(F))=c_1(p)H^{-2}\Psi(H),\quad \mu_p(\sigma(F))=c_2(p)H^2\Psi(H)\mu_p(\sigma_2(F)), \tag{15}$$

where $c_i(p)>0\,(i=1,2)$ are the constants dependent on p. Fix the vector $\mathbf{b}_1=(H,a_{n-1}\dots,a_3,a_2)$ and denote the subclass of $\mathcal{F}_n$ of functions with the same vector $\mathbf{b}_1$ by $\mathcal{F}_n(\mathbf{b}_1)$. The number of different sets $\mathcal{F}_n(\mathbf{b}_1)$ is $\ll H^{n-2}$.

Now for $F\in\mathcal{F}_n(\mathbf{b}_1)$ we estimate $|F(x)|_p$ where $x\in\sigma_2(F)$. From definition of $\sigma_2(F)$ it follows that $|F'(\alpha_F)(x-\alpha_F)|_p<H^{-2}$. For $m\geqslant 2$ by (14) we have

$$|\frac{1}{n!}F^{(n)}(\alpha_F)(x-\alpha_F)^n|_p<(H^{-2}H^{1+v})^n\leqslant H^{-2+2v}.$$

By Taylor's formula and the previous estimates we get

$$|F'(x)|_p\ll H^{-2+2v} \tag{16}$$

for any $x\in\sigma_2(F)$ and $v\leqslant 1/2$.

Further we use essential and inessential domains introduced by Sprindzuk [4]. The domain $\sigma_2(F)$ is called *inessential* if there is a function $\tilde F\in\mathcal{F}_n(\mathbf{b}_1)$ (with $\tilde F\neq F$) such that

$$\mu_p(\sigma_2(F)\bigcap\sigma_2(\tilde F))\geqslant\frac{\mu_p(\sigma_2(F))}{2},$$

and *essential* otherwise.

First, the inessential domains are investigated. Let the domain $\sigma_2(F)$ is inessential and $K_1=\sigma_2(F)\bigcap\sigma_2(\tilde F)$. Then

$$\mu_p(K_1)\geqslant\frac{1}{2}\mu_p(\sigma_2(F))=c_3(p)H^{-2}|F'(\alpha_F)|_p^{-1},$$

where $c_3(p)$ is a constant dependent on p.

Consider the new function $R(x) = F(x) - \tilde{F}(x) = b_1x + b_0$, where $F(x), \tilde{F}(x) \in \mathcal{F}_n(\mathbf{b}_1)$ and $\max(|b_0|, |b_1|) \leqslant 2H$.

By (16),we obtain

$$|R(x)|_p \ll H^{-2+2v} \tag{17}$$

for any $x \in \sigma_2(F)$.

If we assume that $b_1 = 0$ then $|b_0|_p \ll H^{-2+2v}$, but it is contradicted to $|b_0|_p \geqslant |b_0|^{-1} \gg H^{-1}$. Hence, $b_1 \neq 0$. By (17),we have

$$|x + b_0/b_1|_p \ll H^{-2+2v}|b_1|_p^{-1}. \tag{18}$$

Let $K_2 = \{x \in K_0 : \textit{the inequality (18) holds }\}$. Then $K_1 \subseteq K_2$ and $\mu_p(K_2) = c_4(p)H^{-2+2v}|b_1|_p^{-1}$. We have

$$H^{-2+2v}|F'(\alpha_F)|_p^{-1} \ll \mu_p(K_1) \leqslant \mu_p(K_2) \ll H^{-2+2v}|b_1|_p^{-1}.$$

Hence, by (14) and previous estimate we obtain $|b_1|_p \ll |F'(\alpha_F)|_p \ll H^{-1/2}$. From (17) we obtain that $|b_0|_p \ll H^{-1/2}$. Suppose that $s \in \mathbb{Z}^+$ is defined by the inequalities $p^s \leqslant H < p^{s+1}$. For sufficiently large H we obtain that $H^{1/2} \asymp p^{[s/2]}$. Hence $b_1 \asymp p^{[s/2]}b_{11}$ and $b_0 \asymp p^{[s/2]}b_{01}$ for $b_{11}, b_{01} \in \mathbb{Z}$. We have $b_1x+b_0 \asymp p^{[s/2]}(b_{11}x+b_{01})$, where $\max|b_{11}|, |b_{01}| \ll H^{1/2}$. Let $R_1(x) = b_{11}x + b_{01}$, then $H(R_1) \ll H^{1/2}$. Then by (17) we obtain

$$|b_{11}x+b_{01}|_p \ll p^{[s/2]}H^{-2+2v} \ll H^{-3/2+2v} = H(R_1)^{-3+4v} \ll H(R_1)^{-2-\delta},\ \delta > 0\ for\ v < \frac{1}{4}.$$

By Khintchine's Theorem in $\mathbb{Q}_p$ [4], we obtain that the set of x belonging to infinitely many dicks $\sigma_2(F)$ has zero measure.

Now let $\sigma_2(F)$ be essential domain. By the property of p-adic valuation every point $x \in K_0$ belong to no more than one essential domain. Hence

$$\sum_{F\in\mathcal{F}_n(\mathbf{b}_1)} \mu_p(\sigma_2(F)) \ll \mu_p(K_0).$$

From this, (15) and the fact that the number of vectors $\mathbf{b}_1$ is $\ll H^{n-2}$, we have

$$\sum_{\mathbf{b}_1}\sum_{F\in\mathcal{F}_n(\mathbf{b}_1)} \mu_p(\sigma(F)) \ll H^n\Psi(H)\mu_p(K_0).$$

Finally, we obtain

$$\sum_{H=1}^{\infty}\sum_{\mathbf{b}_1}\sum_{F\in\mathcal{F}_n(\mathbf{b}_1)} \mu_p(\sigma(F)) < \infty.$$

Thus, by the Borel–Cantelli Lemma, the set of points x which belong to infinitely many essential domains is of measure zero.

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