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STRUCTURE of the BEST DIOPHANTINE APPROXIMATIONS and MULTIDIMENSIONAL GENERALIZATIONS of the CONTINUED FRACTION [1]

A. D. Bruno (t. Moscow)

Аннотация

Let in a three-dimensional real space two forms be given: a linear form and a quadratic one which is a product of two complex conjugate linear forms. Their root sets are a plane and a straight line correspondingly. We assume that the line does not lie in the plane. Voronoi (1896) and author (2006) proposed two different algorithms for computation of integer points giving the best approximations to roots of these two forms. The both algorithms are one–way: the Voronoi algorithms is directed to the plane and the authors algorithms is directed to the line.

Here we propose an algorithm, which works in both directions. We give also a survey of results on such approach to simultaneous Diophantine approximations in any dimensions.

1. Statement of the problem

Let in $\mathbb{R}^3$ with coordinates $X = (x_1, x_2, x_3)$ the linear form

$$l_1(X) = \langle J, X\rangle \stackrel{\text{def}}{=} j_1x_1 + j_2x_2 + j_3x_3$$

and the quadratic form $l_2(X) = \langle K, X\rangle\langle \overline{K}, X\rangle$ be given. Product

$$f(X) = l_1(X)l_2(X).$$

The root set $f(X) = 0$ is $\mathcal{L}_1 \cup \mathcal{L}_2$ where the plane $\mathcal{L}_1 = \{X : \langle J, X\rangle = 0\}$, the line $\mathcal{L}_2 = \{X : \langle \Re K, X\rangle = \langle \Im K, X\rangle = 0\}$. Put

$$m_i(X) = |l_i(X)|, \quad i = 1, 2; \qquad M(X) = (m_1(X), m_2(X)).$$

The integer point $X \in \mathbb{Z}^3$ is *the best approximation to* $\mathcal{L}_1 \cup \mathcal{L}_2$ if a point $Y \in \mathbb{Z}^3$, $Y \neq 0$ with

$$M(Y) \leqslant M(X), \qquad \|M(Y)\| < \|M(X)\|$$

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is absent.

Problem. *To find algorithm for computation of best approximations.*

Now there are two algorithms solving the problem: by G. Voronoi (1896) [1], it tends to $\mathcal{L}_1$, and by A. Bruno and V. Parusnikov (2006) [2], it tends to $\mathcal{L}_2$. Here we propose new algorithm working in both directions: to $\mathcal{L}_1$ and to $\mathcal{L}_2$.

2. The principal construction

The vector-function

$$M(X) = (m_1(X), m_2(X)) = (|l_1(X)|, |l_2(X)|)$$

maps $\mathbb{R}^3$ into the first quadrant S_+ of the plane $S = \mathbb{R}^2 \ni (m_1, m_2)$. Let $\mathbf{Z}$ be the image of $\mathbb{Z}^3$ except $X = 0$:

$$\mathbf{Z} = M(\mathbb{Z}^3 \setminus 0).$$

$\mathbf{M}$ is the convex hull of $\mathbf{Z}$, and $\partial\mathbf{M}$ is the boundary of $\mathbf{M}$.

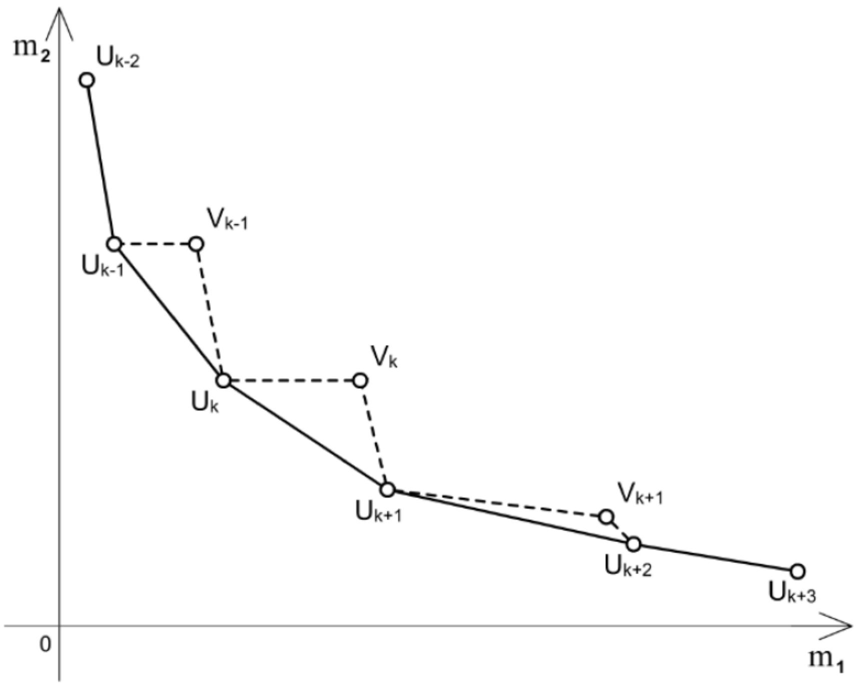

Рис. 1: The convex open polygon $\partial\mathbf{M}$ and additional points V_k.

In consists of vertices and edges. All its vertices are images of the integer points $X \in \mathbb{Z}^3$. Some such images can be at edges. All points of $\partial\mathbf{M} \cap \mathbf{Z}$ are images of the best approximations. So computing the polygon $\partial\mathbf{M}$ is enough for solving our Problem.

For two points $U = (u_1, u_2)$ and $V = (v_1, v_2) \in S_+$ the function

$$\zeta_1(U, V) = \frac{v_2 - u_2}{u_2(u_1 - v_1)},$$

is the value m_1^{-1} in the point $(m_1, 0) \in S_+$ of the axis m_1 where the axis intersects the line going through points U and V.

3. Algorithm

Let we have the basis B_1, B_2, $B_3 \in \mathbb{Z}^3$, $\det(B_1^*, B_2^*, B_3^*) = \pm 1$, ordered in some manner. We compute the next basis $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{B}_3$ by the linear transformation

$$\begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & a \\ 1 & b & c \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix},$$

where a, b, $c \in \mathbb{Z}$; $|a|$, $|b|$, $|c| \leqslant \kappa_l$, differences $M(\tilde{B}_2) - M(B_3)$, $M(\tilde{B}_3) - M(B_3)$ belong either to the III quadrant or to I and IV quadrants (Fig. 2);

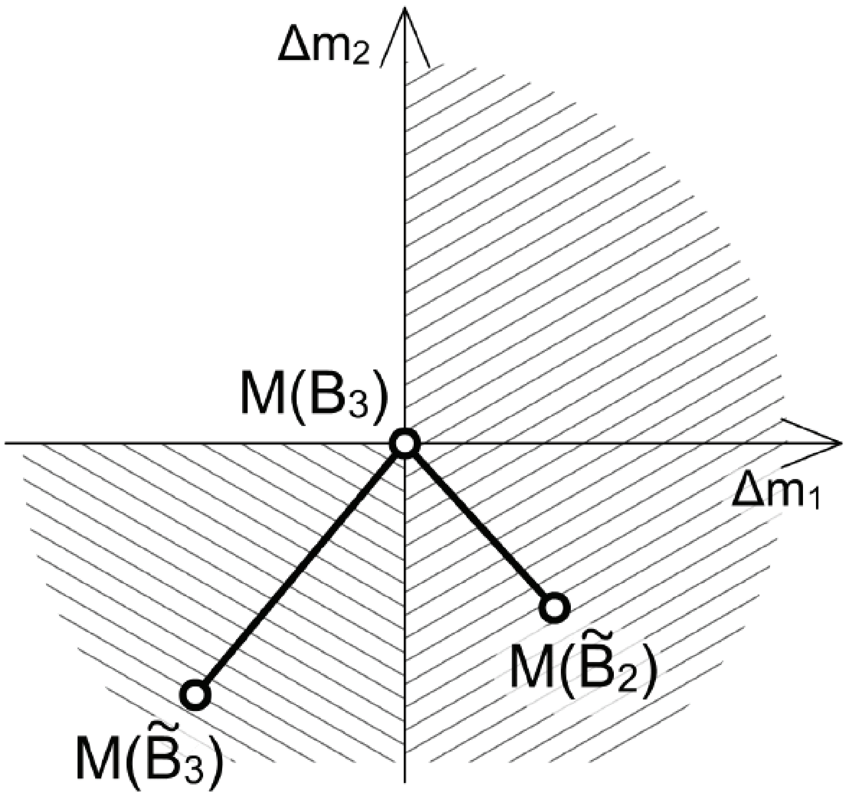

Рис. 2: Places for images of new basis points.

$$\tilde{B}_2 : \ \zeta_1(M(B_3), M(\tilde{B}_2)) = \max_a \zeta_1(M(B_3), M(\tilde{B}_2));$$

$$\tilde{B}_3 : \ \zeta_1(M(B_3), M(\tilde{B}_3)) = \max_{b,c} \zeta_1(M(B_3), M(\tilde{B}_3)).$$

Among all possible values of $\tilde{B}_2$ and $\tilde{B}_3$ we choose those giving biggest inclinations for lines going through points $M(\tilde{B}_2)$, $M(\tilde{B}_3)$ and the point $M(B_3)$.

Here $\kappa_l = 3 \cdot 2^l$ is the movable boundary for $|a|$, $|b|$, $|c|$: if one of $|a|$, $|b|$, $|c|$ reaches κ_l, then we replace κ_l by κ_{l+1}, and repeat computation of $\tilde{B}_2$ and $\tilde{B}_3$. To the basis $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{B}_3$ we again apply the described algorithm and so on (Fig. 1). The algorithm works in both directions: to $\mathcal{L}_1$ and to $\mathcal{L}_2$, enough to permute forms $l_1(X)$ and $l_2(X)$.

4. Algebraic case

Let the polynomial $P(\lambda) \stackrel{\text{def}}{=} \lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma$ have integral coefficients and negative discriminant. It has 3 roots $\lambda_1 \in \mathbb{R}$, $\lambda_2 = \overline{\lambda_3} \in \mathbb{C}$. Put $J = (1, \lambda_1, \lambda_1^2)$, $K = (1, \lambda_2, \lambda_2^2)$, $l_1(X) = \langle J, X\rangle$, $l_2(X) = \langle K, X\rangle\langle \overline{K}, X\rangle$. According to the Dirichlet Theorem, the field $\mathbb{Q}(\lambda_1)$ has one fundamental unity which corresponds to the unimodular substitution $X = DY$. The substitution is the automorphism of $|f(X)| = |l_1(X)||l_2(X)|$ and of the open polygon $\partial\mathbf{M}$. Thus, the polygon $\partial\mathbf{M}$ is periodic.

So our algorithm allows to find the minimal period of $\partial\mathbf{M}$ and the corresponding fundamental unit of the field.

The algorithm was implemented and was checked on a lot of polynomials [3].

5. Three linear forms and positive discriminant

The similar approach in the case of three linear forms leads to consideration of a polyhedral surface $\partial\mathbf{M}$ in the 3-dimensional first octant $(m_1, m_2, m_3) \geqslant 0$. In the algebraic case the polynomial $P(\lambda)$ must have the positive discriminant then three real roots of $P(\lambda)$ give 3 linear forms. The surface $\partial\mathbf{M}$ has 2 independent periods corresponding to 2 fundamental units. The algorithm was described early (2005–2007 [4, 5, 6]).

6. General situation

In $\mathbb{R}^n$ there are given l linear forms $f_i(X)$, $i = 1, \ldots, l$ and k quadratic forms $f_j(X)$, $j = l+1, \ldots, l+k$ and $l + 2k = n$. The map

$$m_i = |f_i(X)|, \qquad i = 1, \ldots, l+k$$

transforms $\mathbb{Z}^n \setminus 0$ in the set $\mathbf{Z} \subset \mathbb{R}_+^{l+k}$. $\mathbf{M} \subset \mathbb{R}_+^{l+k}$ is the convex hull of $\mathbf{Z}$. Boundary $\partial\mathbf{M}$ has dimension $l + k - 1$.

In algebraic case $\partial\mathbf{M}$ has $l + k - 1$ independent periods [7]. They can be found by computation of $\partial\mathbf{M}$ by a similar algorithm.

7. Comparison with other approaches

In 1895–1896 F. Klein [8], H. Minkowski [9] and G. Voronoi [1] proposed 3 different approaches to generalization of the continued fraction for the case of 3 linear forms in $\mathbb{R}^3$. Our approach is nearer to the Voronoi's one, but different from it.

The Klein's approach was proposed again independently by B. Scubenko (1988) [10] and by V. Arnold (1998) [11]. The term "Klein's polyhedra" I introduced (1994) [12] as reaction on the term "Arnold's polyhedra" introduced by G. Lachand (1993) [13]. We found (1994–2002) [12, 14, 15, 16, 17] that Klein's polyhedra cannot give a background for an algorithm generalizing the continued fraction. So I proposed (2005) [4] one polyhedron **M** which is in a sense the convex hull of 8 Klein's polyhedra. Nevertheless now several groups in different countries study the Klein's polyhedra.

The Minkowski's approach was developed for n linear forms in $\mathbb{R}^n$ by J. Lagarias (1994) [18]. But his algorithm is essentially more complicated than algorithm proposed by the author.

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Институт прикладной математики им. М. В. Келдыша РАН
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